Three Factor Model for Convertible Bond

The owner of a convertible bond (CB) receives periodic coupon payments from the issuer, but can also convert the CB into the issuer's stock. The convertible bond may also include call and put provisions, which respectively allow the issuer to buy back the convertible bond and the owner to put the convertible bond for respective preset amounts.

We present a three factor trinomial tree based model for pricing the CB, where the three factors are

- the short interest rate corresponding to the bond's coupon currency,
- the issuer's stock price, and
- the exchange rate from the issuer's stock currency into the bond's coupon currency.

Here the short interest rate process is assumed to be of Ho-Lee form under the bond's coupon currency risk-neutral probability measure, and the stock price and foreign exchange rate processes are assumed to follow geometric Brownian motion with drift under their respective risk-neutral probability measures.

The stock price process is then expressed under the bond's coupon currency risk neutral probability measure by means of a quanto adjustment. Under the bond's coupon currency risk neutral probability measure, then, the short interest rate, stock price and foreign exchange rate processes respectively follow geometric Brownian motion with drift, but are driven by pair-wise correlated Brownian motions.

We next define three related random variables, which are each taken to be particular linear combinations of the original short interest rate, stock price and foreign exchange rate random variables. Here the respective linear combinations are chosen such that the processes for the new random variables are now driven by pairwise uncorrelated Brownian motions.

Assume that the stock price process , $\left\{S_t \;\middle|\; t \geq 0\right\}$, satisfies under its respective risk neutral probability measure, a stochastic differential equation (SDE) of the form

$$dS_t = S_t\left(\left[r_S - q_S\right]dt + \sigma_S d\tilde{W}_t^S\right), \quad t \geq 0,$$

where

* $\left\{\tilde{W}_t^S \;\middle|\; t \geq 0\right\}$ denotes standard Brownian motion,
* r_S denotes the short term interest rate,
* q_S denotes the dividend yield, and
* σ_S denotes the volatility.

Suppose that the exchange rate process, $\left\{C_t \;\middle|\; t \geq 0\right\}$, from one unit of stock currency into the corresponding units of bond coupon currency, satisfies, under the bond's coupon currency risk neutral probability measure, a SDE of the form

$$dC_t = C_t\left(\left[R_t - r_S\right]dt + \sigma_C dW_t^C\right), \quad t \geq 0,$$

where

* $\left\{W_t^C \;\middle|\; t \geq 0\right\}$ denotes standard Brownian motion,
* R_t denotes the short term interest rate, and
* σ_C denotes the volatility.

Assume also that short term interest rate process, $\{R_t \mid t \ge 0\}$, corresponding to the bond's coupon currency satisfies, under the bond's coupon currency risk neutral probability measure, a Ho-Lee type SDE of the form

$$dR_t = \theta(t)dt + \sigma_R dW_t^R, \qquad t \ge 0,$$

where

* $\{W_t^R \mid t \ge 0\}$ denotes standard Brownian motion,
* σ_R is the volatility, and
* $\theta(t)$ is chosen to match the initial term structure of zero coupon bond prices.

Then, under the bond's coupon currency risk neutral probability measure,

$$dS_t = S_t\left(\left[r_S - q_S - \rho_{CS}\sigma_S\sigma_C\right]dt + \sigma_S dW_t^S\right), \quad t \ge 0,$$

where

* $\{W_t^S \mid t \ge 0\}$ denotes standard Brownian motion.

Here we let ρ_{CR}, ρ_{CS} and ρ_{SR} denote the respective correlation coefficients between the Brownian motions

* $\{W_t^R \mid t \ge 0\}$ and $\{W_t^C \mid t \ge 0\}$,
* $\{W_t^S \mid t \ge 0\}$ and $\{W_t^C \mid t \ge 0\}$, and
* $\{W_t^S \mid t \ge 0\}$ and $\{W_t^R \mid t \ge 0\}$.

We then have

$$\begin{cases} d\log C_t = \mu_C dt + \sigma_C dW_t^C, & t \geq 0, \\ d\log S_t = \mu_S dt + \sigma_S dW_t^S, & t \geq 0, \\ dR_t = \mu_R dt + \sigma_R dW_t^R, & t \geq 0, \end{cases}$$

where

* $\mu_S = r_S - q_S - \rho_{CS}\sigma_S\sigma_C - \dfrac{\sigma_S^2}{2}$,

* $\mu_C = R_t - r_S - \dfrac{\sigma_C^2}{2}$, and

* $\mu_R = \theta(t)$.

Next let

$$\begin{aligned} E_t &= \log\left(S_t C_t\right), \\ &= \log S_t + \log C_t. \end{aligned}$$

Then, from Ito's Lemma,

$$dE_t = \mu_E dt + \sigma_S dW_t^S + \sigma_C dW_t^C, \quad t > 0,$$

where $\mu_E = \mu_S + \mu_C$. Next let

$$\begin{aligned} \sigma_E &= \sqrt{\frac{1}{t} Var\left(\sigma_S W_t^S + \sigma_C W_t^C\right)}, \\ &= \left(\sigma_S^2 + 2\rho_{CS}\sigma_S\sigma_C + \sigma_C^2\right)^{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} \rho_{ER} &= corr\left(\sigma_S W_t^S + \sigma_C W_t^C,\ W_t^R\right), \\ &= \frac{\sigma_S \rho_{RS} + \sigma_C \rho_{CR}}{\sigma_E}; \end{aligned}$$

furthermore let $h = \dfrac{\rho_{RE}\sigma_E}{\sigma_R}$.

Next let

$$\begin{aligned} X_t &= E_t - hR_t \\ &= \log S_t + \log C_t - hR_t , \end{aligned}$$

and let Y_t be of the form

$$\begin{aligned} Y_t &= \alpha R_t + \beta E_t + \gamma \log C_t , \\ &= \alpha R_t + \beta\left(\log S_t + \log C_t\right) + \gamma \log C_t , \end{aligned}$$

where α, β and γ are finite and real. From Ito's Lemma,

$$\begin{cases} dR_t = \mu_R dt + \sigma_R dW_t^R , & t \geq 0, \\ dX_t = \mu_X dt + \sigma_S dW_t^S + \sigma_C dW_t^C - h\sigma_R dW_t^R , & t \geq 0, \\ dY_t = \mu_Y dt + \alpha\sigma_R dW_t^R + \beta\left(\sigma_S dW_t^S + \sigma_C dW_t^C\right) + \gamma\sigma_C dW_t^C , & t \geq 0, \end{cases}$$

where

* $\mu_R = \theta(t)$,
* $\mu_X = \mu_E - h\mu_R$, and
* $\mu_Y = \alpha\mu_R + \beta\mu_E + \gamma\mu_C$.

Here

$$\begin{cases} X_0 = \log S_0 + \log C_0 - hR_0 , \\ Y_0 = \alpha R_0 + \beta \log S_0 + \left(\beta + \gamma\right)\log C_0 . \end{cases}$$

Let $P(0,t)$ denote the price at time zero, in the bond's coupon currency, of a zero coupon bond with face value of 1 (bond's coupon currency) unit and maturity of t. We then approximate the drift term μ_E by

$$\tilde{\mu}_E = \tilde{R}_t - q_S - \frac{1}{2}\left(\sigma_C^2 + \rho_{CS}\sigma_S\sigma_C + \sigma_S^2\right),$$

where

$$e^{\int_0^t \tilde{R}_s ds} = \frac{1}{P(0,t)}.$$

Next we assume that the short term interest rate, r_S, corresponding to the stock currency is deterministic.

We next construct respective trinomial trees to approximate the processes for each new random variable, but with each tree based on the same time slice partition of the CB tenor. A new tree, which combines these respective trees, is then defined such that the set of nodes on a particular time slice of the combined tree equals the cross product of the set of nodes in each of the three respective trees at this time slice.

Each node on the combined tree then has $3^3 = 27$ children; and, since the processes for the new random variables are driven by pairwise independent Brownian motions, the probability of branching to any of these children nodes is given by the product of the corresponding probabilities on each of the individual trees. The values for the original short interest rate, stock price and foreign exchange rate are now obtained, at each node on the combined tree, by inverting a related linear system of equations.

At each tree time slice we keep track of the call strike, put strike and stock conversion level at this time; we note, however, that FP does not include accrued interest in the respective call and put strike levels at tree times where a coupon is not paid. FP assumes that the CB owner can choose to hold, put or convert the bond into stock. The CB issuer can then choose to call the bond. If the bond is called, however, the owner can then force the bond to be converted into stock.

The CB is then valued by traversing the combined tree using backward induction. At each tree node we apply the hold, put or call logic above, discounting by the corresponding stochastic interest rate.

We note that FP constructs an individual trinomial tree by first defining a related binomial tree. A trinomial tree approximation is then constructed by coalescing adjacent pairs of binomial tree time slices into corresponding single trinomial tree steps. Here the trinomial tree branching probabilities are taken as particular products of the probabilities on the two corresponding binomial tree time slices.

We note that, while this operation matches the mean of the random variable at the new trinomial tree time slice, its variance is only approximately matched. We further note that FP's spreadsheet implementation restricts the maximum number of trinomial tree time slices to be less than 47. With this restriction, then, it may not be possible to select the maximum trinomial tree inter time slice spacing sufficiently small such that branching probabilities on the trinomial tree are all positive and less than one. The resulting tree, then, may not be stable.

Since the bond issuer may default, the CB value is affected by credit risk. FP models this effect by reducing the coupon amounts paid by the underlying bond in the CB valuation. You can find more about convertible bond valuation at https://finpricing.com/lib/EqConvertible.html